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Weighted digraphs having exactly two nonzero skew eigenvalues

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Abstract

The matrix with respect to the real n-vector $\mathbf{a} = [a_1, a_2, \ldots, a_n]^T$ is an $n \times n$ matrix $M(\mathbf{a}) = [m_{ij}]$, where $m_{ij} = a_i - a_j$. The weighted digraph $D_{\mathbf{a}}$ with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ indexed with the vector $\mathbf{a}$ is obtained by drawing an arc of weight $a_i - a_j$ from v_i to v_j if $a_i - a_j > 0$. If the vector $\mathbf{a}$ has at least two distinct elements, then $M(\mathbf{a})$ has exactly two non zero eigenvalues. Further we discuss the structural properties of $D_{\mathbf{a}}$.

Keywords: Digraph, skew matrix, eigenvalues.

2020 MSC: 05C50, 05C20.

1. Introduction

The eigenvalues of the adjacency matrices of digraphs are studied in [3, 5, 6, 10] and eigenvalues of the skew-adjacency matrices of digraphs are studied in [1, 4, 7, 8, 9, 12]. In this paper we consider a special kind of matrix generated from the real vector and corresponding weighted digraph and study their spectral and structural properties. The concept degree subtraction matrix considered in [11] becomes the special case of the results of this paper.

Let D be a digraph with vertex set $V(D) = \{v_1, v_2, \ldots, v_n\}$ and arc set $\Gamma(D)$. Let n be the number of vertices and m be the number of arcs of D. The number of arcs leaving from (respectively terminating at) the vertex v is called the *outdegree* (respectively *indegree*) of v and is denoted by $od_D(v)$ (respectively $id_D(v)$). A directed graph is said to be *connected* if every pair of vertices of D is joined by some path. If every pair of vertices has a arc then D is called a *complete digraph*. A directed graph D is called *bipartite* if its vertex set $V(D)$ can be partitioned into two sets V_1 and V_2 such that every arc of D has one end (starting or terminating) in V_1 and other end (terminating or starting) in V_2. A bipartite digraph D is called *complete bipartite* if every vertex of V_1 is joined to all vertices of V_2. A directed graph D is called k-*partite* if its vertex set $V(D)$ can be partitioned into k sets $V_1, V_2, \ldots, V_k$ such that every arc of D has one end in V_i and other end in V_j, $i \neq j$. A k-partite digraph D is called *complete* k-*partite* if every vertex of V_i is joined to all vertices of V_j, $i \neq j$. A directed path P_k on k vertices $v_1, v_2, \ldots, v_k$ in D is called *unidirected*

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path if all its arcs have same direction. That is, if $v_1, v_2, \ldots, v_k$ are the vertices of P_k, then P_k is unidirected if there is an arc from v_i to v_{i+1}, $i = 1, 2, \ldots, k-1$. A directed cycle C_k is said to be *unidirected cycle* if all its arcs are oriented in clockwise direction or all in anticlockwise direction. If the arcs of the digraph have some weight then it is called a *weighted digraph*. For graph theoretic terminology we refer the book [2].

2. Eigenvalues of a matrix $M(\mathbf{a})$

Let $\mathbf{a} = [a_1, a_2, \ldots, a_n]^T$ be the real n-vector. The matrix with respect to $\mathbf{a}$ is defined as $n \times n$ matrix $M(\mathbf{a}) = [m_{ij}]$, where $m_{ij} = a_i - a_j$.

Consider a vector $\mathbf{a} = [5, -4, 5, 2, 3]^T$. Therefore

$$M(\mathbf{a}) = \begin{bmatrix} 0 & 9 & 0 & 3 & 2 \\ -9 & 0 & -9 & -6 & -7 \\ 0 & 9 & 0 & 3 & 2 \\ -3 & 6 & -3 & 0 & -1 \\ -2 & 7 & -2 & 1 & 0 \end{bmatrix} \tag{2.1}$$

and eigenvalues of $M(\mathbf{a})$ are $\mathbf{i}\sqrt{274}$, 0, 0, 0 and $-\mathbf{i}\sqrt{274}$, where $\mathbf{i} = \sqrt{-1}$.

Theorem 2.1. *Let $\mathbf{a} = [a_1, \ldots, a_n]^T$ be the real n-vector and $\mathbf{e} = [1, 1, \ldots, 1]^T$ be the all one n-vector. Let $M(\mathbf{a})$ be the matrix with respect to the vector $\mathbf{a}$. Then*

(i) *$M(\mathbf{a})$ is skew symmetric.*
(ii) *if $\mathbf{a}$ and $\mathbf{e}$ are linearly dependent then $M(\mathbf{a})$ is a zero matrix.*
(iii) *if $\mathbf{a}$ and $\mathbf{e}$ are linearly independent then $\mathrm{rank}(M(\mathbf{a})) = 2$.*
(iv) *the eigenvalues of $M(\mathbf{a})$ are $\pm\mu\mathbf{i}$ and 0 ($n-2$ times), where $\mu = \sqrt{\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i - a_j)^2}$.*

Proof.

(i) Note that $M(\mathbf{a}) = \mathbf{a}\mathbf{e}^T - \mathbf{e}\mathbf{a}^T$.
Hence $M(\mathbf{a})^T = \mathbf{e}\mathbf{a}^T - \mathbf{a}\mathbf{e}^T = -M(\mathbf{a})$. So the matrix $M(\mathbf{a})$ is skew symmetric.
(ii) Let $\mathbf{a} = \lambda\mathbf{e}$. Therefore $M(\mathbf{a}) = \mathbf{a}\mathbf{e}^T - \mathbf{e}\mathbf{a}^T = \lambda\mathbf{e}\mathbf{e}^T - \lambda\mathbf{e}\mathbf{e}^T = 0$.
(iii) Note that $M(\mathbf{a}) = \begin{bmatrix} \mathbf{a} & -\mathbf{e} \end{bmatrix} \begin{bmatrix} \mathbf{e}^T \\ \mathbf{a}^T \end{bmatrix}$. Since $\mathbf{a}$ and $\mathbf{e}$ are linearly independent, $\begin{bmatrix} \mathbf{a} & -\mathbf{e} \end{bmatrix}$ is of rank 2. Therefore $M(\mathbf{a})$ has rank 2.
(iv) Since $M(\mathbf{a})$ is a real skew symmetric, its eigenvalues appear in conjugate pairs. However $M(\mathbf{a})$ has at most rank 2, so the eigenvalues of $M(\mathbf{a})$ are $\pm\mu\mathbf{i}$ and 0 ($n-2$ times). Consequently,

$$-2\mu^2 = \mathrm{trace}[(M(\mathbf{a}))^2] = \mathrm{trace}[M(\mathbf{a})M(\mathbf{a})^T] = -\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i - a_j)^2.$$

That is

$$\mu = \sqrt{\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i - a_j)^2}.$$

□

Corollary 2.2. *If there exists at least two distinct elements in the real n-vector $\mathbf{a} = [a_1, a_2, \ldots, a_n]^T$, then the matrix $M(\mathbf{a})$ has exactly two nonzero eigenvalues which are $\pm\,\mathbf{i}\sqrt{\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i - a_j)^2}$, where $\mathbf{i} = \sqrt{-1}$.*

Proof. Since the vector **a** contains at least two distinct elements, by Cauchy-Schwartz inequality

$$\left(\sum_{i=1}^{n} a_i\right)^2 < n\sum_{i=1}^{n} a_i^2.$$

This implies that

$$\sqrt{\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i - a_j)^2} = \sqrt{n\sum_{i=1}^{n} a_i^2 - \left(\sum_{i=1}^{n} a_i\right)^2} \neq 0.$$

Therefore by Theorem 2.1(iv), the result follows. □

3. Weighted digraphs corresponding to the matrix $M(\mathbf{a})$

Following algorithm gives the weighted digraph having exactly two nonzero skew eigenvalues from the real n-vector $\mathbf{a} = [a_1, a_2, \ldots, a_n]^T$, in which at least two elements are distinct.

Algorithm: Construction of digraph $D_{\mathbf{a}}$ with respect to real n-vector $\mathbf{a} = [a_1, a_2, \ldots, a_n]^T$.

Start

1. Assign the vertex v_i, corresponding to the element a_i of the vector **a** for $i = 1, 2, \ldots, n$.
2. If $a_i - a_j > 0$ then draw an arc with weight $a_i - a_j$ from the vertex v_i to the vertex v_j, for all $i, j = 1, 2, \ldots, n$.
3. The resulting graph is a weighted digraph.

Stop

The digraph with respect to the vector $\mathbf{a} = [5, -4, 5, 2, 3]^T$ constructed using above algorithm is shown in Fig. 1. Its matrix is in Eq. (2.1).

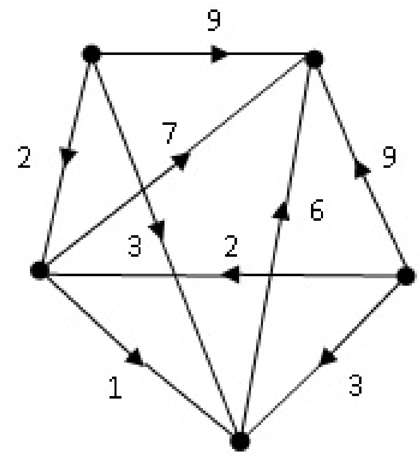

Figure 1: Digraph with respect to the vector $\mathbf{a} = [5, -4, 5, 2, 3]^T$.

Remark 3.1. If all a_i's of a vector $\mathbf{a}$ are equal then the resulting digraph $D_{\mathbf{a}}$ has no arcs.

Remark 3.2. Let v_i be the vertex of a weighted digraph $D_{\mathbf{a}}$ corresponding to the element a_i of $\mathbf{a}$. If a_i is greater than k elements in $\mathbf{a}$ then $od_D(v_i) = k$ and if a_i is smaller than k elements in $\mathbf{a}$ then $id_D(v_i) = k$.

Theorem 3.3. *The weighted digraph $D_{\mathbf{a}}$ with respect to the vector $\mathbf{a} = [a_1, a_2, \dots, a_n]^T$ has no unidirected cycle.*

Proof. Let C_p be the directed cycle in $D_{\mathbf{a}}$ of length p. If $p = 1$ and 2, the result is trivial. Suppose $p \geqslant 3$. Let $v_1, v_2, \dots, v_n$ be the vertices of $D_{\mathbf{a}}$ corresponding to the elements $a_1, a_2, \dots, a_n$ of $\mathbf{a}$. Without loss of generality consider $a_1 > a_2 > \cdots > a_p$ for $p = 3, 4, \dots, n$. Therefore $a_i - a_j > 0$ if $i > j$, for $i, j = 1, 2, \dots, p$. Hence there is a unidirected path with vertices $v_1, v_2, \dots, v_p$ where $(v_i, v_{i+1}) \in \Gamma(D)$, $i = 1, 2, \dots, p-1$, and since $a_1 > a_p$, there is no arc from v_p to v_1. Thus the vertices $v_1, v_2, \dots, v_p$ do not induces a unidirected cycle C_p in $D_{\mathbf{a}}$. □

Theorem 3.4. *The weighted digraph $D_{\mathbf{a}}$ with respect to the vector $\mathbf{a} = [a_1, a_2, \dots, a_n]^T$ is complete if and only if all the elements of $\mathbf{a}$ are distinct.*

Proof. Suppose all elements of $\mathbf{a}$ are not distinct. Then there exist at least one pair of elements in $\mathbf{a}$ whose difference is zero. Hence corresponding vertices are not joined by any arc in $D_{\mathbf{a}}$, a contradiction to the fact that $D_{\mathbf{a}}$ is complete. Therefore all the elements of $\mathbf{a}$ are distinct.

Conversely, let $v_1, v_2, \dots, v_n$ be the vertices of a weighted digraph $D_{\mathbf{a}}$ corresponding to the elements of the vector $\mathbf{a}$. Since each pair of elements of $\mathbf{a}$ has nonzero difference, there is an arc between every pair of vertices of $D_{\mathbf{a}}$. Hence the digraph $D_{\mathbf{a}}$ is complete graph. □

Theorem 3.5. *The weighted digraph $D_{\mathbf{a}}$ with respect to the vector $\mathbf{a} = [a_1, a_2, \dots, a_n]^T$ is connected if and only if for each element a_i of $\mathbf{a}$, there exists at least one element a_j in $\mathbf{a}$, $i \neq j$, such that $a_i \neq a_j$.*

Proof. Let v_i be the vertex of a weighted digraph corresponding to the element a_i of the vector $\mathbf{a}$. Suppose $D_{\mathbf{a}}$ is connected. Then there exist a path between every pair of vertices. Hence each vertex is adjacent to at least one vertex. Thus for every element a_i of $\mathbf{a}$, there exists atleast one element a_j in $\mathbf{a}$, $i \neq j$, such that $a_i \neq a_j$.

Conversely, suppose for each element a_i of $\mathbf{a}$, there exists at least one element a_j in $\mathbf{a}$, $i \neq j$, such that $a_i \neq a_j$. Therefore for each vertex v_i, there exists at least one vertex v_j, adjacent to v_i. Thus every pair of vertices of $D_{\mathbf{a}}$ is joined by some path. Hence the digraph $D_{\mathbf{a}}$ is connected. □

Theorem 3.6. *The weighted digraph $D_{\mathbf{a}}$ with respect to the vector $\mathbf{a} = [a_1, a_2, \dots, a_n]^T$ is complete k-partite if and only if*

$$\begin{array}{ccccccccc} a_1 & = & a_2 & = & \cdots & = & a_{p_1} & = & q_1 \\ a_{p_1+1} & = & a_{p_1+2} & = & \cdots & = & a_{p_2} & = & q_2 \\ a_{p_2+1} & = & a_{p_2+2} & = & \cdots & = & a_{p_3} & = & q_3 \\ \vdots & & & & & & & & \\ a_{p_{k-1}+1} & = & a_{p_{k-1}+2} & = & \cdots & = & a_{p_k} & = & q_k \end{array}$$

where $q_i \neq q_j$ for $i \neq j$, $i, j = 1, 2, \dots, k$ and $p_1 + p_2 + \cdots + p_k = n$.

Proof. Suppose the digraph $D_{\mathbf{a}}$ is complete k-partite. The vertex set of $D_{\mathbf{a}}$ can be partitioned into k sets $V_1, V_2, \dots, V_k$ such that each vertex of V_i is joined to all vertices of V_j, $i \neq j$ and no two vertices in any V_i are adjacenct. Let the vertices of V_{l+1} be indexed with the elements of the set $A_{l+1} = \{a_{p_l+1}, a_{p_l+2}, \dots, a_{p_{l+1}}\}$, $l = 0, 1, \dots, k-1$ and $p_0 = 0$. If any two elements of A_{l+1} are not equal then the corresponding vertices in V_{l+1} are joined by an arc, a contradiction. Hence all the elements of A_{l+1} are equal. Further, since each vertex of V_i is adjacent to all vetrices of V_j, the elements of A_i are not equal to any element of A_j, $i \neq j$, $i, j = 1, 2, \dots, k$.

Conversely, let $A_{l+1} = \{a_{p_l+1}, a_{p_l+2}, \dots, a_{p_{l+1}}\}$, $l = 0, 1, \dots, k-1$ and $p_0 = 0$, where $a_{p_l+1} = a_{p_l+2} = \cdots = a_{p_{l+1}} = q_l$. Let V_{l+1} be the partite set of the vertex set of weighted digraph $D_{\mathbf{a}}$ corresponding to A_{l+1}. Therefore every arc of $D_{\mathbf{a}}$ has one end in V_i and other end in V_j, $i \neq j$, $i, j = 1, 2, \dots, k$. Hence the digraph $D_{\mathbf{a}}$ is complete k-partite. □

References

[1] C. Adiga, R. Balakrishnan, W. So, *The skew energy of a digraph*, Linear Algebra Appl., **432** (2010), 1825–1835. 1

[2] M. Behzad, G. Chartrand, L. Lisniak-Foster, *Graphs and Digraphs*, Wadsworth International Group, Belmont, (1979). 1

[3] R. A. Brualdi, *Spectra of digraphs*, Linear Algebra Appl., **432** (2010), 2181–2213. 1

[4] M. Cavers, S. M. Cioabă, S. Fallat, D. A. Gregory, W. H. Haemers, S. J. Krikland, J. J. McDonald, M. Tsatsomeros, *Skew-adjacency matrices of graphs*, Linear Algebra Appl., **436** (2012), 4512–4529. 1

[5] S. Furtado, *Comparison on the spectral radii of weighted digraphs that differ in a certain subdigraph*, Electron. Notes Discrete Math., **54** (2016), 85–90. 1

[6] C. D. Godsil, *Eigenvalues of graphs and digraphs*, Linear Algebra Appl., **46** (1982), 43–50. 1

[7] Y. Hou, T. Lei, *Characteristic polynomial of skew-adjacency matrices of oriented graphs*, Elect. J. Combin., **18** (2011), # p156. 1

[8] X. Li, H. Lian, *A survey on the skew energy of oriented graphs*, arXiv:1304.5707. 1

[9] H. Lian, X. Li, *Skew-spectra and skew energy of various products of graphs*, Trans. Comb., **4** (2015), 13–21. 1

[10] H. Lin, J. Shu, Y. Wu, G. Yu, *Spectral radius of strongly connected digraphs*, Discrete Math., **312** (2012), 3663–3669. 1

[11] H. S. Ramane, K. C. Nandeesh, G. A. Gudodagi, B. Zhou, *Degree subtraction eigenvalues and energy of graphs*, Comput. Sci. J. Moldova, **26** (2018), 146–162. 1

[12] H. S. Ramane, K. C. Nandeesh, I. Gutman, X. Li, *Skew equienergetic digraphs*, Trans. Comb., **5** (2016), 15–23. 1